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Steady currents induced by oscillations round islands

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An oscillating current such as a tidal stream or an inertial oscillation may have a horizontal scale of the order of many times the local depth of water. Thus an island projecting from an otherwise uniform sea bed will give rise to a local, periodic disturbance near the island. It is shown that this disturbance may be resolved into two waves travelling in opposite senses round the island. If the particle orbits at large distances are circular, then only one of these waves has non-zero amplitude.

In addition to the oscillatory motion, however, there is a steady d.c. streaming, or mass-transport velocity, whose magnitude is of order $u^2/\sigma a$ where u denotes the magnitude of the oscillatory velocity at large distances, σ denotes the radian frequency, and a is the radius of the island. In this paper the profile of the streaming velocity is calculated for circular islands, with or without shoaling regions offshore. It is shown that resonance with the free modes trapped by the shoaling regions may greatly increase the streaming velocity. Viscosity (or horizontal mixing) also tends to increase the streaming velocity close to the shoreline.

The conclusions are supported by some simple model experiments. It is suggested that such streaming may partly account for the observed pattern of currents near Bermuda.

1. Introduction

The best-known example of a steady, rectified flow associated with an oscillatory motion is the mass-transport velocity in a progressive water wave, first discovered by Stokes (1847); (see also Longuet-Higgins 1953, 1960). It has been pointed out by Longuet-Higgins (1969*b*) that significant rectified flows are also to be expected in many types of long-period, oscillatory ocean currents, particularly in tides, internal waves and motions depending on bottom topography, such as double Kelvin waves.†

The aim of the present paper is to consider another topographic effect, the effect of an island which projects from the sea bed in an otherwise uniform, oscillating current.

Assuming the displacement in the initial oscillation to be small compared to

† Mass-transport in tidal flows was first considered by Hunt & Johns (1963). Pedlosky (1965), Robinson (1965, ch. 17, pp. 504–533) and Munk & Moore (1968) have suggested mass-transport effects in other types of current, but for a criticism of their analysis see Moore (1969).

the radius of the island, the first problem to be considered is the linear response of the fluid near the island to a forced oscillation in the ocean at large distances. This problem is solved in §2 below, for islands with vertical sides. For a circular island it is possible to evaluate the local disturbance caused by the island in very simple terms. Particularly, when the currents at large distances are inertial, it turns out that the disturbance is seen locally as a wave progressing anticlockwise round the island, in the northern hemisphere; the tangential velocity at the boundary is just double what it would be in the absence of the island.

The second-order currents associated with this flow are considered in §3. It is shown that the rectified flow outside a thin viscous boundary layer consists of a current circulating *anticlockwise* round the island and falling off rapidly with radial distance, like the inverse fifth power of the radius at first. But inside a thin viscous boundary layer, the mass-transport current is reinforced by viscosity. In fact the effect of viscosity is to multiply the non-viscous streaming velocity just beyond the boundary layer by a factor $\frac{5}{2}$ independent of ν. This effect is analogous to the effect of the viscous layer on the bottom in a progressive water wave (Longuet-Higgins 1953), experimentally verified by Allen & Gibson (1959) and others. After some time, the vorticity introduced by the viscous boundary layer diffuses outwards throughout the interior of the fluid, augmenting the initial circulation.

Some simple experiments to test this theoretical prediction are described in §4 of the present paper.

So far the discussion has referred to islands with vertical sides. On the other hand, when the island is surrounded by a shallow region or 'skirt', the possibility arises of free waves becoming trapped near the island, as was shown by Rhines (1969) in a particular case. Moreover, the amplitude of the forced oscillations may increase greatly near the resonant frequencies. In §5 we calculate both the amplitude of the forced oscillations and the magnitude of the steady mass-transport associated with them. Figure 6 is a typical plot of the relative magnification near the resonant frequencies. A simple model experiment is described in §6, in which oscillations were set up in a rotating basin containing islands of various shapes. Strong currents were observed to be circulating round the islands, consistent with the above predictions.

The possible connection of this phenomenon with the observed pattern of currents near Bermuda is discussed in §7.

2. Forced oscillations round a cylindrical island

We imagine an island with vertical sides, in an ocean of locally uniform depth h, in which there exists a large-scale system of oscillating currents. The latter may be inertial oscillations, tidal waves, planetary waves or other types of large-scale oscillation, and may be generated, for example, by tidal or atmospheric forces. The purpose of this section is to calculate the local effect of the island on this large-scale system of currents and in particular to find the form of the surface elevation at the edge of the island itself.

We shall suppose that the differential equation governing the surface elevation

ζ is the classical equation for long waves of small amplitude oscillating harmonically with radian frequency σ, in an ocean of uniform depth h and constant Coriolis parameter f, that is to say

$$\left(\nabla^2 + \frac{\sigma^2 - f^2}{gh}\right)\zeta = 0 \tag{2.1}$$

(see Lamb 1932). The associated current-vector $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{-g}{\sigma^2 - f^2}(i\sigma\nabla\zeta - \mathbf{f}\wedge\nabla\zeta), \tag{2.2}$$

where $\mathbf{f}$ denotes the vertical vector of magnitude f. We assume that $\sigma^2 \neq f^2$, in general, but when it is appropriate we shall take the solutions to the limit as $\sigma \to \pm f$, bearing in mind that the limit may be singular.

Now for most applications, and certainly with islands of the dimensions of order 100 km the scale of the local disturbance will be exceedingly small compared to $\sqrt{(gh)}/f$. Since σ is assumed to be of the same order as f, it follows that in (2.1) the second term is negligible for practical purposes, and that ζ satisfies simply Laplace's equation,

$$\nabla^2\zeta = 0. \tag{2.3}$$

In the present context (2.1) may be called the exact differential equation for ζ and (2.3) the approximate equation. A solution of the exact equation does exist representing waves trapped round a circular island (Longuet-Higgins 1969*a*). We shall find solutions to the approximate equation, and show that they do tend, in one case of special interest, to the corresponding exact solution.

Suppose first that we have a circular island of radius a. Let the rectangular components (u, v) of the current at large distances r from the centre of the island be given by

$$u = A\,e^{-i\sigma t}, \quad v = B\,e^{-i\sigma t}, \tag{2.4}$$

where A and B are complex constants. It is understood that the real part of the expressions on the right is to be taken. The radial and tangential components of velocity are then given by

$$\left.\begin{aligned} u_r &= u\cos\theta + v\sin\theta = C\,e^{i(\theta-\sigma t)} + D\,e^{-i(\theta+\sigma t)}, \\ u_\theta &= -u\sin\theta + v\cos\theta = iC\,e^{i(\theta-\sigma t)} - iD\,e^{-i(\theta+\sigma t)}, \end{aligned}\right\} \tag{2.5}$$

where $C = \frac{1}{2}(A - iB)$ and $D = \frac{1}{2}(A + iB)$. In other words, the motion can be formally separated into two component waves, one with complex amplitude C, rotating clockwise round the island (when $\sigma < 0$), the other with complex amplitude D, rotating anticlockwise.

To find the corresponding surface elevation we take (2.2) in the form,

$$\left.\begin{aligned} (\partial/\partial r)(g\zeta) &= i\sigma u_r + f u_\theta, \\ \frac{1}{r}(\partial/\partial\theta)(g\zeta) &= i\sigma u_\theta - f u_r, \end{aligned}\right\} \tag{2.6}$$

and assume that ζ vanishes at the origin. It follows that the surface elevation ζ_0 in the absence of the island is given uniquely by

$$g\zeta_0 = i(\sigma + f)\,Cr\,e^{i(\theta-\sigma t)} + i(\sigma - f)\,Dr\,e^{-i(\theta+\sigma t)}. \tag{2.7}$$

In inertial oscillations, ζ_0 vanishes. Such motion can be represented either by $C = 0$ and $\sigma = f$ or alternatively by $D = 0$ and $\sigma = -f$.

Now to take account of the presence of the island we add to (2.7) a surface elevation ζ_1 of the form,

$$g\zeta_1 = P(a^2/r)\, e^{i(\theta-\sigma t)} + Q(a^2/r)\, e^{-i(\theta+\sigma t)} \tag{2.8}$$

in which the constants P and Q are to be chosen so as to make the normal component of velocity vanish at the circumference $r = a$. Thus $\zeta = \zeta_0 + \zeta_1$ must satisfy

$$\left(i\sigma\frac{\partial\zeta}{\partial r} - \frac{f}{r}\frac{\partial\zeta}{\partial\theta}\right)_{r=a} = 0. \tag{2.9}$$

Hence
$$-(\sigma^2 - f^2)\, C - i(\sigma+f)\, P = 0, \tag{2.10}$$

with a similar equation for Q. Solutions of these equations are

$$P = -i(\sigma - f)\, C, \quad Q = -i(\sigma+f)\, D. \tag{2.11}$$

Therefore altogether we have

$$g\zeta = i[(\sigma+f)\, r - (\sigma-f)\,(a^2/r)]\, C\, e^{i(\theta-\sigma t)} + i[(\sigma-f)\, r + (\sigma+f)\,(a^2/r)]\, D\, e^{-i(\theta+\sigma t)}. \tag{2.12}$$

At the perimeter of the island this reduces to

$$g\zeta = 2ifa[C\, e^{i(\theta-\sigma t)} - D\, e^{-i(\theta+\sigma t)}]. \tag{2.13}$$

Thus if u_∞ denotes the transverse component of the velocity at infinity, the surface elevation at the edge of the island is simply given by

$$g\zeta = 2afu_\infty. \tag{2.14}$$

In the important case of inertial oscillations we take $D = 0$ and $\sigma \to -f$ in (2.12), giving

$$g\zeta = 2ifC(a^2/r)\, e^{i(\theta-\sigma t)}. \tag{2.15}$$

Thus, *for inertial motions, the surface elevation progresses round the island in the clockwise sense, in the northern hemisphere.* Since in this case ζ_0 vanishes, the surface elevation is given by ζ_1, alone.

To find the components of velocity (u_r, u_θ) near the island, some care must be taken. The exact expressions

$$\left.\begin{aligned} u_r &= \frac{-1}{\sigma^2 - f^2}\left(i\sigma\frac{\partial}{\partial r} - \frac{f}{r}\frac{\partial}{\partial\theta}\right) g\zeta, \\ u_\theta &= \frac{-1}{\sigma^2 - f^2}\left(\frac{i\sigma}{r}\frac{\partial}{\partial\theta} + f\frac{\partial}{\partial r}\right) g\zeta \end{aligned}\right\} \tag{2.16}$$

may be used only when $\sigma^2 \neq f^2$. However, if we substitute in (2.16) from (2.12), we obtain in the general case,

$$\left.\begin{aligned} u_r &= \left(1 - \frac{a^2}{r^2}\right)[C\, e^{i(\theta-\sigma t)} + D\, e^{-i(\theta+\sigma t)}], \\ u_\theta &= i\left(1 + \frac{a^2}{r}\right)[C\, e^{i(\theta-\sigma t)} - D\, e^{-i(\theta+\sigma t)}]. \end{aligned}\right\} \tag{2.17}$$

It is remarkable that this solution is formally independent of the Coriolis parameter f and of the frequency σ. To find a solution for inertial motions we again let $D = 0$ and $\sigma = -f$ giving

$$\left.\begin{aligned} u_r &= \left(1-\frac{a^2}{r^2}\right) C\, e^{i(\theta-\sigma t)}, \\ u_\theta &= i\left(1+\frac{a^2}{r^2}\right) C\, e^{i(\theta-\sigma t)}. \end{aligned}\right\} \tag{2.18}$$

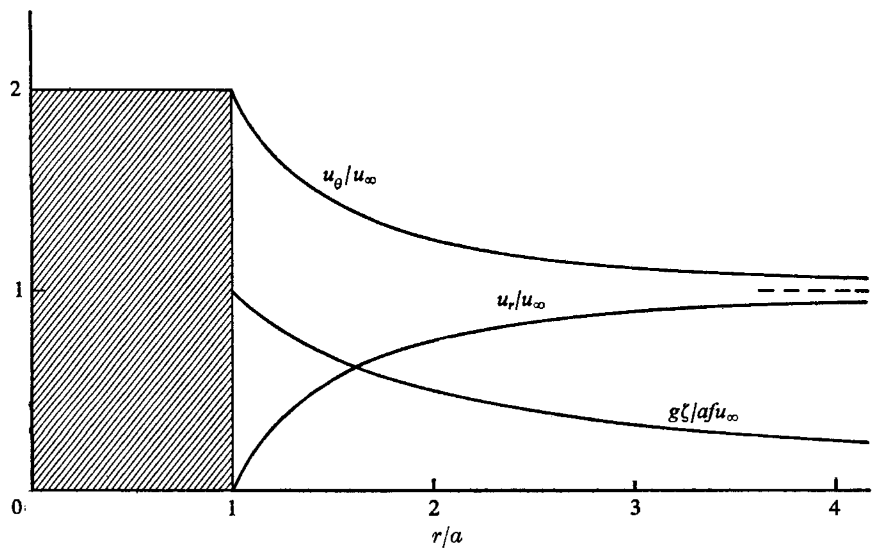

FIGURE 1. Graphs of the radial and tangential components u_r, u_θ of the orbital velocity in an oscillation in the neighbourhood of an island of radius a. The associated surface elevation is also shown.

These expressions for u_r, u_θ and ζ are illustrated in figure 1. At the circumference of the island equations (2.18) become simply

$$u_r = 0, \quad u_\theta = 2iC\, e^{i(\theta-\sigma t)}. \tag{2.19}$$

Comparing this with (2.5) we see that the presence of the island exactly doubles the transverse component of velocity at the perimeter.

It may be noted that (2.15) is also the first term in the asymptotic expansion of the *exact* solution of (2.1), namely,

$$g\zeta = 2ifC\, ka^2 K_1(k_r)\, e^{i(\theta-\sigma t)}, \tag{2.20}$$

where
$$k = \left(\frac{f^2-\sigma^2}{gh}\right)^{\frac{1}{2}} \quad (\sigma^2 < f^2). \tag{2.21}$$

This represents a mode trapped exponentially at large distances from the island (Longuet-Higgins 1969*a*). Equation (2.15) is the limit of (2.20) when $af/\sqrt{(gh)} \to 0$. The expressions for the currents, (2.18), also correspond to (2.7) of the paper just referred to, if the constants in the factors $(1 \pm a^2/r^2)$ are replaced by terms varying only logarithmically with r.

Because in the present approximation the surface elevation ζ satisfies Laplace's equation (2.3) it is simple to extend these results to islands of other shape, by a conformal transformation. The boundary condition $\mathbf{u}.\mathbf{n} = 0$ is transformed into the same condition at all non-singular points of the boundary, by (2.2). For example, the exterior of the elliptic island,

$$\frac{x_2}{a^2}+\frac{y^2}{b^2} = 1 \quad (a > b > 0), \tag{2.22}$$

having principal axes a and b, is transformed by the substitutior.

$$\left.\begin{aligned} z \equiv x+iy &= \tfrac{1}{2}c\left(\lambda+\frac{1}{\lambda}\right), \\ \lambda &= \mu+i\nu, \\ c &= (a^2-b^2)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.23}$$

into the exterior of the circle,

$$\mu^2+\nu^2 = \frac{a+b}{a-b}. \tag{2.24}$$

Moreover, at infinite distance from the island, $z \sim \frac{1}{4}c\lambda$. So the corresponding expression for ζ is of the same form as for the circular island, but in terms of λ, not z. To obtain the corresponding expressions in terms of x and y we replace λ by that solution of the quadratic equation,

$$\lambda^2+(2/c)\,z\lambda+1 = 0, \tag{2.25}$$

which tends to infinity as $z \to \infty$. The expressions for the currents follow from the formula,

$$(u-iv) = \mathscr{R}\left\{\frac{-g}{\sigma^2-f^2}(i\sigma+f)\frac{d\zeta}{d\lambda}\frac{d\lambda}{dz}\right\}. \tag{2.26}$$

From this solution it can be shown that u, v and ζ are in antiphase at diametrically opposite points on the island, and that currents of near-inertial frequency are associated with a surface elevation which travels anticlockwise round the ellipse, as for the circle.

Similar transformations may be devised to cope with islands of more complicated shape. For *more* than one island (the 'archipelago problem'), extended techniques are available from potential theory. But in such a case variations of depth between the islands are likely to prove important also.

As suggested earlier (1969*a*), it is possible to understand the energy in the frequency-band centred on 0·73 c.p.d. at Oahu as the local manifestation of near-inertial currents on a wider scale surrounding the island. The observed phase difference of about 130° between Mokuoloe and Honolulu (Mokuoloe leading) can be interpreted as due to the tendency of the surface elevation to progress round the island in the clockwise direction. Although the angle subtended at the centre of the island by Mokuoloe and Honolulu is only about 40°, the difference may be explained by the tendency of some energy to be guided round the elongated ridge on which Oahu stands.

It should be stressed that motions of near-inertial frequency are not necessarily uniform currents, as is often supposed. The inertial frequency is the limiting frequency for many other types of motion, including internal waves and motions trapped by topography and β-effect. The possibility that they are internal waves cannot yet be dismissed. An interesting experimental program awaits those who wish to investigate the matter further.

3. Mass-transport currents

Let $\mathbf{u}$ denote any time-periodic velocity field such that to the first order in the amplitude the mean velocity $\bar{\mathbf{u}}$ vanishes. If the (second-order) mean velocity of a marked particle (the Lagrangian mean velocity) be denoted by $\bar{\mathbf{u}}_L$, and the (second-order) mean velocity at a fixed position (the Eulerian mean velocity) be denoted by $\bar{\mathbf{u}}_E$, then it can be shown (see Longuet-Higgins 1953) that

$$\bar{\mathbf{u}}_L = \bar{\mathbf{u}}_E + \mathbf{U}, \tag{3.1}$$

where

$$\mathbf{U} = \overline{\int \mathbf{u}\, dt \,.\, \nabla \mathbf{u}}. \tag{3.2}$$

The quantity $\mathbf{U}$ has been called the Stokes velocity (Longuet-Higgins 1969*b*) after G. G. Stokes, who first evaluated this term for surface waves on water.

When the motion is practically two-dimensional, as in the present application, the two components U, V of the Stokes velocity may be expressed in terms of a stream-function ψ_s (cf. Longuet-Higgins 1953) as follows:

$$U = \frac{\partial \psi_s}{\partial y}, \quad V = -\frac{\partial \psi_s}{\partial x}, \tag{3.3}$$

where

$$\psi_s \equiv \overline{u \int v\, dt} \equiv -\overline{\int u\, dt\, v}. \tag{3.4}$$

In the radial co-ordinates suited to the present problem, the corresponding expressions are

$$U_r = \frac{1}{r}\frac{\partial \psi_s}{\partial \theta}, \quad U_\theta = -\frac{1}{r}\frac{\partial \psi_s}{\partial r}, \tag{3.5}$$

where

$$\psi_s \equiv \overline{u_r \int u_\theta dt} \equiv -\overline{\int u_r dt\, u_\theta}. \tag{3.6}$$

For the simple progressive motion given by (2.18) we have

$$\Psi = \frac{1}{2\sigma}\left(1 - \frac{a^2}{r^4}\right) CC^*, \tag{3.7}$$

where a star denotes the complex conjugate quantity. From (3.5) it follows that $U_r = 0$, as we should expect from symmetry, while

$$U_\theta = \frac{2a^5}{r^5}\frac{CC^*}{a\sigma}. \tag{3.8}$$

Thus the Stokes velocity is altogether tangential and falls off like the inverse fifth power of the radial distance r. At the circumference of the island ($r = a$) we have

$$U_\theta = 2CC^*/a\sigma. \tag{3.9}$$

To find either $\mathbf{u}_L$ or $\mathbf{u}_E$ we must employ the full dynamical equations for the mean motion. The motion is in effect two-dimensional and hence by analogy with surface waves on water (Longuet-Higgins 1953, §4), or otherwise, the differential equation for the stream function ψ_E of the Eulerian mean flow is

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\psi_E = 0. \tag{3.10}$$

Initially, if the motion is irrotational, we expect that $\psi_E \equiv 0$ everywhere, that is to say

$$\psi_L = \psi_s = \overline{u \int v\, dt}, \tag{3.11}$$

except near the boundary of the fluid. The subsequent development of the motion represents the diffusion of velocity outwards from the circumference of the island.

Near the boundary $r = a$, we expect a boundary layer of the type described in Longuet-Higgins (1953, §7). It is shown there that if the first-order normal and tangential velocities are given by q_n and q_s respectively then just beyond the boundary layer there is a tangential velocity given by

$$u_L = \frac{5i+3}{4\sigma} q_s \frac{\partial q_s^*}{\partial s} \tag{3.12}$$

(*loc. cit.* equation (189)), the co-ordinate s being measured tangentially to the boundary. In the present problem, we replace q_s by u_θ and s by $a\theta$. Thus

$$u_L = \frac{5i+3}{4a\sigma} u_\theta \frac{\partial u_\theta^*}{\partial \theta}. \tag{3.13}$$

For the progressive wave described by (2.19) we find then

$$u_L = 5CC^*/a\sigma, \tag{3.14}$$

which is to be compared to the non-viscous value given by (3.9). In other words, the presence of a viscous boundary layer increases the velocity near the boundary† in the ratio 5/2.

Is a final steady state possible? Writing $\partial/\partial t = 0$ in (3.10) we find that ψ_E has to satisfy the biharmonic equation in two dimensions, and therefore

$$\psi = Pr^2 \log r + Qr^2 + R \log r + S, \tag{3.15}$$

where P, Q, R, S are arbitrary constants. If u_E is to vanish at infinity, P and Q must both be zero. The value of S is immaterial, and if the boundary condition at $r = a$ is to be satisfied we must have

$$R = 3CC^*/\sigma, \tag{3.16}$$

† These paradoxical effects have been well verified by experiments in the case of water waves; see Russell & Osorio (1958); Allen & Gibson (1959). In the case of progressive waves it has been found experimentally that equation (3.12) is valid also for turbulent flows. A theoretical justification, (based on the assumption that the eddy viscosity is some function of the distance from the boundary) is given in an appendix to the paper by Russell & Osorio (1958).

so that the transverse component u_L of the Lagrangian mean velocity is given ultimately by

$$u_L = \left(\frac{2a^5}{r^5}+\frac{3a}{r}\right)\frac{CC^*}{a\sigma}. \tag{3.17}$$

More generally, when the motion consists of two progressive waves travelling in opposite directions (see (2.17)) then substitution in (3.6) gives for the Stokes velocity

$$\psi_s = \frac{a^4}{2\sigma r^4}(CC^*-DD^*)+\frac{1}{2\sigma}\left(\frac{a^4}{r^4}-1\right)(CD^*\,e^{2i\theta}-C^*D\,e^{-2i\theta}). \tag{3.18}$$

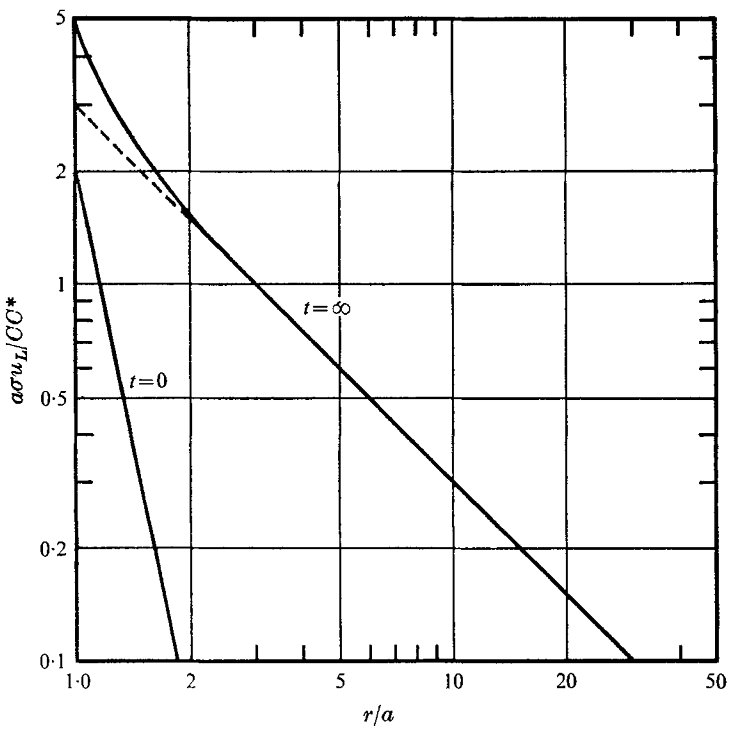

FIGURE 2. Graphs of the initial and final Lagrangian mean velocities u_L round an island of radius a, as a function of r/a.

When the amplitudes of the two waves are equal ($CC^* = DD^*$), the first term vanishes, and taking the origin of θ and t, so that C and D are real and equal, we have

$$\psi_s = \frac{i}{\sigma}\left(\frac{a^4}{r^4}-1\right)C^2\sin 2\theta. \tag{3.19}$$

This represents a kind of dipole motion.

Similarly, (3.13) gives in general for the velocity just outside the boundary layer:

$$u_L = \frac{5}{a\sigma}(CC^*-DD^*)+\frac{3}{a\sigma}(CD^*\,e^{2i\theta}-C^*D\,e^{-2i\theta}). \tag{3.20}$$

When, as before, C and D are real and equal,

$$u_L = \frac{3C^2}{a\sigma}\sin 2\theta. \tag{3.21}$$

The result is similar to the streaming first found by Schlichting (1932) to occur in the presence of an oscillating cylinder. See also Batchelor (1967, ch. 5). The fluid enters the boundary layer at $\theta = 0°$ and $180°$ in the plane of oscillation and leaves it at the intermediate points $\theta = \pm 90°$.

It will be noticed that we have neglected friction on the bottom and taken into account only the friction at the vertical sides of the island. This may be justified if the vertical mixing is small compared to the horizontal mixing.

4. Experimental verification (i)

So long as the flow is practically two-dimensional and non-divergent, the mass transport currents depend only on the relative motion between the axis of the cylinder and the fluid at infinite distance. Hence these currents should be the same as if the fluid at infinity were stationary and the island were made to oscillate in a horizontal plane.

The latter arrangement is the more convenient experimentally. Accordingly, the author constructed a mechanism (shown in figure 3(*a*), plate 1), whereby a cylindrical can of radius $a = 3$ in. and length 9 in. could be made to oscillate so that its axis described a smaller vertical cylinder of radius $b = \frac{1}{2}$ in. Throughout the oscillation the orientation of the can remained fixed. This was achieved by fixing the can† to a rectangular frame pivoted on four joints, each of which was made to oscillate in parallel by four gear wheels driven from a central vertical shaft. This vertical shaft was driven by a bevel gear attached to a horizontal shaft, which in turn could be operated either by hand or by attachment to an electric motor.

The whole apparatus was then supported on two angle pieces laid across the top of a circular tank of diameter 30 in., as in figure 3(*b*), plate 1. The tank was filled with water to within 1 in. of the top of the can. With a syringe, some dilute ink was injected into the water close to the surface of the can and near its centre. The can was then made to oscillate by turning the handle as shown, or with an electric motor. The time t required for the ink to make a complete circuit was measured with a stop-watch, and also the mean period T of the first ten oscillations.

According to §3 above, the magnitude $|C|$ of the relative velocity between the fluid at infinity and the axis of the cylinder is equal to σb. Hence the streaming velocity just outside the oscillatory boundary layer is equal to $5\sigma b^2/a$ by (3.14). It follows that the time taken for a marked particle of fluid to complete a circuit of the cylinder, just outside the oscillatory boundary layer, is just equal to $a^2/5b^2$ periods of the oscillation. This result is independent not only of the viscosity but also of the period of oscillation, within the range of the present approximations.

† Kindly supplied by my wife.

Corresponding to the apparatus described above we have $a/b = 6$; hence the time required for a circuit is $6^2 \div 5$, that is 7·2, periods.

This simple result in practice is slightly complicated by the following considerations:

(i) The radius of the outer cylinder is finite. This introduces errors of order a^2/A^2 where A is the radius of the outer cylinder.

(ii) The surface is free. This introduces errors of order $\sigma^2 a/g$, where g denotes the acceleration of gravity.

(iii) The amplitude of the motion is finite. This introduces relative errors of order b/a.

(iv) The thickness of the oscillatory boundary layer, though small, is finite. This introduces errors of order δ/a, where $\delta = (\nu/\sigma)^{\frac{1}{2}}$.

(v) Outside the boundary layer the motion is not steady until a time of order $(r-a)^2/\nu$ after starting.

The presence of these errors, especially (iii), leads us to expect discrepancies of the order of 10 % between the observations and the simple theory of § 3.

Nevertheless, the experiment was tried. On starting the motion from rest by running the motor so that the mean period T of the first ten oscillations lay between 1·37 and 6·91 sec, it was found that the time taken for the first trace of dye to orbit the can lay always between 5·4 and 8·8 periods of oscillation with a mean of 7·3 periods (see table 1). The agreement was thus at least as good as expected.

T (sec)	t (sec)	N $(= t/T)$	Serial number of observation
1·37	7·4	5·4	3
1·81	12·3	6·8	2
2·58	20·5	7·9	1
3·52	24·2	6·9	4
4·65	37·8	8·1	5
	28·9	6·2	6
5·30	43·5	8·2	8
	38·5	7·3	9
6·91	60·4	8·8	7

TABLE 1. The observed number N of cycles taken by a particle to make a complete circuit of the cylinder in figure 3

5. Islands with sloping sides: free oscillations

So long as the sides of the island are vertical it is impossible for free waves to be trapped near the island unless the horizontal dimensions of the ocean are at least of order $\sqrt{(gh)}/f$ (see Longuet-Higgins 1969*a*). This restricts the practical possibility of wave trapping with vertical sides to baroclinic motions.

The situation is quite different if the island is surrounded by a sloping shelf or 'skirt'. Then trapping becomes reasonably possible, and a double infinity of free trapped motions appears. An example restricted to the case $\sigma \ll f$ has been studied by Mysak (1967). Rhines (1967, 1969) has considered a more general case when the depth $h(r)$ is assumed to be given by

$$h = \left\{\begin{array}{ll} 0 & (0 < r < a), \\ h_1(r/a)^\alpha & (a < r \leqslant b), \\ h_2 & (b \leqslant r < \infty), \end{array}\right\} \tag{5.1}$$

(see figure 4), α being any positive constant.† When $\alpha < 1$ the sloping 'skirt' is concave upwards; when $\alpha = 1$ the skirt is conical; and when $\alpha > 1$ it is convex. Continuity of h at $r = b$ requires that $h_2 = h_1(b/a)^\alpha$.

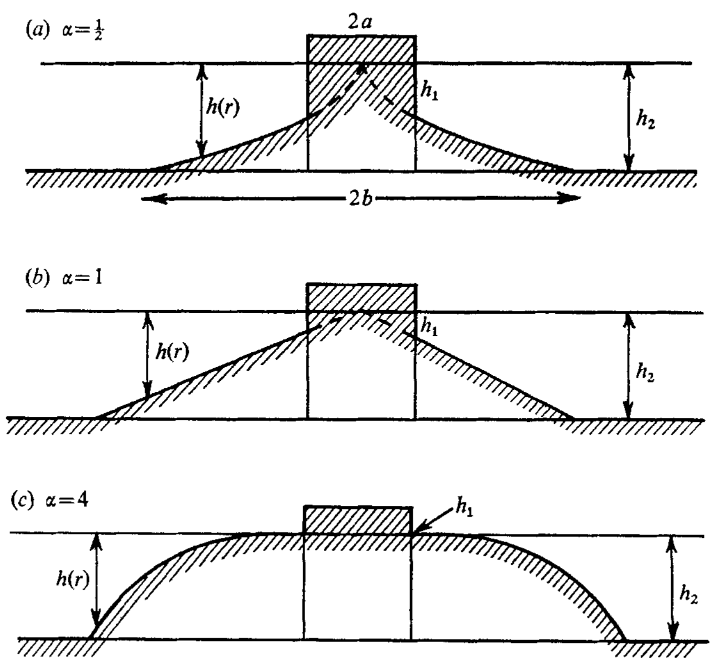

FIGURE 4. Cross-section of the model island with a 'skirt' given by (5.1) in three typical cases: (a) $\alpha = \frac{1}{2}$, (b) $\alpha = 1$, (c) $\alpha = 4$.

Rhines (1969, pp. 195–198) has considered the scattering of a Rossby wave by an island of the above form. Here we shall assume a simple model in which f is constant, and we shall consider simply the response to an oscillation whose form at infinity is given by (2.4) or (2.5). We shall then proceed as in §3 to calculate the corresponding mass transport velocities. Owing to the possibility of resonant excitation of the free modes, the mass-transport can be very greatly amplified.

Neglecting the dynamical effect of the horizontal convergence (which serves only to increase the hydrostatic pressure), we may assume the existence of a stream function ψ such that

$$hu_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad hu_\theta = -\frac{\partial\psi}{\partial r}. \tag{5.2}$$

† Rhines treated especially the case $\alpha = \frac{1}{2}$. Phillips (1966) considered the motion in an annular region between two concentric cylinders, when $\alpha = 2$.

From the conservation of potential vorticity it follows that ψ has to satisfy the differential equation,

$$\frac{\partial}{\partial t}\nabla\cdot\left(\frac{1}{h}\nabla\psi\right)-\mathbf{f}\cdot\nabla\left(\frac{1}{h}\right)\wedge\nabla\psi=0, \tag{5.3}$$

where $\mathbf{f}$ denotes the vector of magnitude f directed vertically. We now seek solutions to this equation in the form,

$$\psi=\hat{\psi}(r)\,e^{i(n\theta-\sigma t)}, \tag{5.4}$$

where n is a positive integer. We are particularly interested in the case $n = 1$. It follows that in the sloping region $a < r < b$, $\hat{\psi}$ must satisfy the ordinary differential equation,

$$\frac{d^2\hat{\psi}}{dr^2}+\frac{1-\alpha}{r}\frac{d\hat{\psi}}{dr}-\frac{n(n+\alpha f/\sigma)}{r^2}\hat{\psi}=0. \tag{5.5}$$

In the constant-depth region $r > b$ we use the same equation but with α set equal to 0. We have further to satisfy the boundary conditions that

$$\left.\begin{aligned}&\hat{\psi}\to 0 && (r\to a),\\ &\hat{\psi},\frac{d\hat{\psi}}{dr}\ \text{continuous} && (r\to b),\\ &\hat{\psi}\sim Cr^n && (r\to\infty).\end{aligned}\right\} \tag{5.6}$$

The differential equation (5.5) is satisfied by taking

$$\hat{\psi}=P_1(r/a)^{p_1}+P_2(r/a)^{p_2}, \tag{5.7}$$

where P_1, P_2 are arbitrary constants and p_1, p_2 are the roots (assumed different) of the quadratic equation,

$$p^2-\alpha p+n(n+\alpha f/\sigma)=0. \tag{5.8}$$

Thus

$$p_1, p_2=\tfrac{1}{2}\pi\pm\beta, \tag{5.9}$$

where

$$\beta=(n^2+n\alpha f/\sigma+\alpha^2/4)^{\frac{1}{2}} \tag{5.10}$$

provided $\beta \neq 0$. The boundary condition at $r = a$ is satisfied by taking

$$P_1=-P_2=P,$$

say. Then we have

$$\hat{\psi}=2P(r/a)^{\alpha/2}\sinh\left[\beta\ln(r/a)\right]\quad(a<r\leqslant b). \tag{5.11}$$

When $r > b$, the differential equation and the boundary condition at infinity are satisfied by

$$\hat{\psi}=Cr^n+Qr^{-n}, \tag{5.12}$$

where Q is a constant to be determined. Now to satisfy the boundary conditions at $r = b$ we must have

$$\left.\begin{aligned}2P(b/a)^{\frac{1}{2}\alpha}\sinh\xi&=Cb^n+Qb^{-n},\\ 2P(b/a)^{\frac{1}{2}\alpha}(\beta\cosh\xi+\tfrac{1}{2}\alpha\sinh\xi)&=nCb^n-nQb^{-n},\end{aligned}\right\} \tag{5.13}$$

where

$$\xi=\beta\ln(b/a). \tag{5.14}$$

Thus the ratios of P, Q and C are given by the two equations

$$\left.\begin{aligned} P(b/a)^{\frac{1}{2}\alpha}[(n+\tfrac{1}{2}\alpha)\sinh\xi+\beta\cosh\xi] &= nCb^n, \\ P(b/a)^{\frac{1}{2}\alpha}[(n-\tfrac{1}{2}\alpha)\sinh\xi-\beta\cosh\xi] &= nQb^{-n}. \end{aligned}\right\} \tag{5.15}$$

The free modes are given by $C = 0$, $P \neq 0$, and so

$$(n+\tfrac{1}{2}\alpha)\sinh\xi+\xi\cosh\xi/\ln(b/a) = 0. \tag{5.16}$$

Since $\ln(b/a) > 0$, there are no real roots ξ; but when β and ξ are imaginary, say $\beta = i\beta'$ and $\xi = i\xi'$, (5.16) becomes

$$\xi'\cot\xi' = -(n+\tfrac{1}{2}\alpha)\ln(b/a). \tag{5.17}$$

We see that for each value of n there is an infinite sequence of possible roots $\xi_1', \xi_2', \ldots$ such that $(m-\frac{1}{2})\pi < \xi_m' < (m+\frac{1}{2})\pi$. Each root ξ_m' corresponds to a possible free mode, and so to a resonant condition. Moreover, from (5.10) we have

$$n^2+n\alpha f/\sigma+\tfrac{1}{4}\alpha^2 = -\beta'^2, \tag{5.18}$$

so that necessarily

$$\sigma/f < 0, \tag{5.19}$$

that is to say the free waves must progress clockwise round the island in the northern hemisphere ($f > 0$) and anticlockwise in the southern hemisphere. From (5.18) we have also

$$-\frac{\sigma}{f} = \frac{\alpha n}{n^2+\frac{1}{2}\alpha^2+\beta'^2} < \frac{\alpha n}{n^2+\frac{1}{4}\alpha^2}. \tag{5.20}$$

The term on the right attains its maximum value 1 when $n = \frac{1}{2}\alpha$. Hence in all cases $|\sigma/f| < 1$, that is to say the frequency of the free modes is always less than the inertial frequency. It may be shown that the corresponding stream function $\hat{\psi}$, given by

$$\hat{\psi} = \left\{\begin{array}{ll} 2iP(r/a)^{\frac{1}{2}\alpha}\sin[\beta'\ln(r/a)] & (a < r < b), \\ Cr^n, & (b < r < \infty), \end{array}\right\} \tag{5.21}$$

has exactly m zeros in the interval $a < r < \infty$, as well as the zero at $r = a$. Outside $r = a$, $|\hat{\psi}|$ decreases monotonically and tends to 0 as $r \to \infty$. The corresponding circulation is in m cells in the radial direction, $2n$ cells in the transverse direction.

The relative frequencies σ/f of the free modes are shown in figure 5 as functions of b/a, for some typical values of the constant α.

We are now in a position to calculate the mass transport velocities in both forced and free oscillations. From (3.6) it follows that the stream function ψ_s for the Stokes velocity is given by

$$\psi_s = -\overline{\frac{1}{r}\frac{\partial\psi}{\partial\theta}\int\frac{\partial\psi}{\partial r}dt}. \tag{5.22}$$

From (5.4) this is

$$\psi_s = \frac{n}{2\sigma r}\hat{\psi}\frac{d\hat{\psi}^*}{dr} = \frac{n}{4\sigma r}\frac{d|\hat{\psi}|^2}{dr}. \tag{5.23}$$

In the free oscillations, $\hat{\psi}$ has in general $(m-1)$ maxima and $(m-1)$ zeros in $a < r < \infty$, so that $|\hat{\psi}|^2$ has $(2m-2)$ maxima. Hence the Stokes velocity $\partial\psi_s/\partial r$ changes sign $(2m-2)$ times in $a < r < \infty$.

In the forced oscillations, we can calculate the Stokes velocity from (5.23). In particular, at the perimeter of the island, where $\hat{\psi}$ vanishes we have from (5.23)

$$U_s = \frac{n}{2\sigma a}\left|\frac{d\hat{\psi}}{dr}\right|^2 = \frac{2n\beta^2P^2}{\sigma a^3}, \tag{5.24}$$

β and P being given by (5.10) and (5.14).

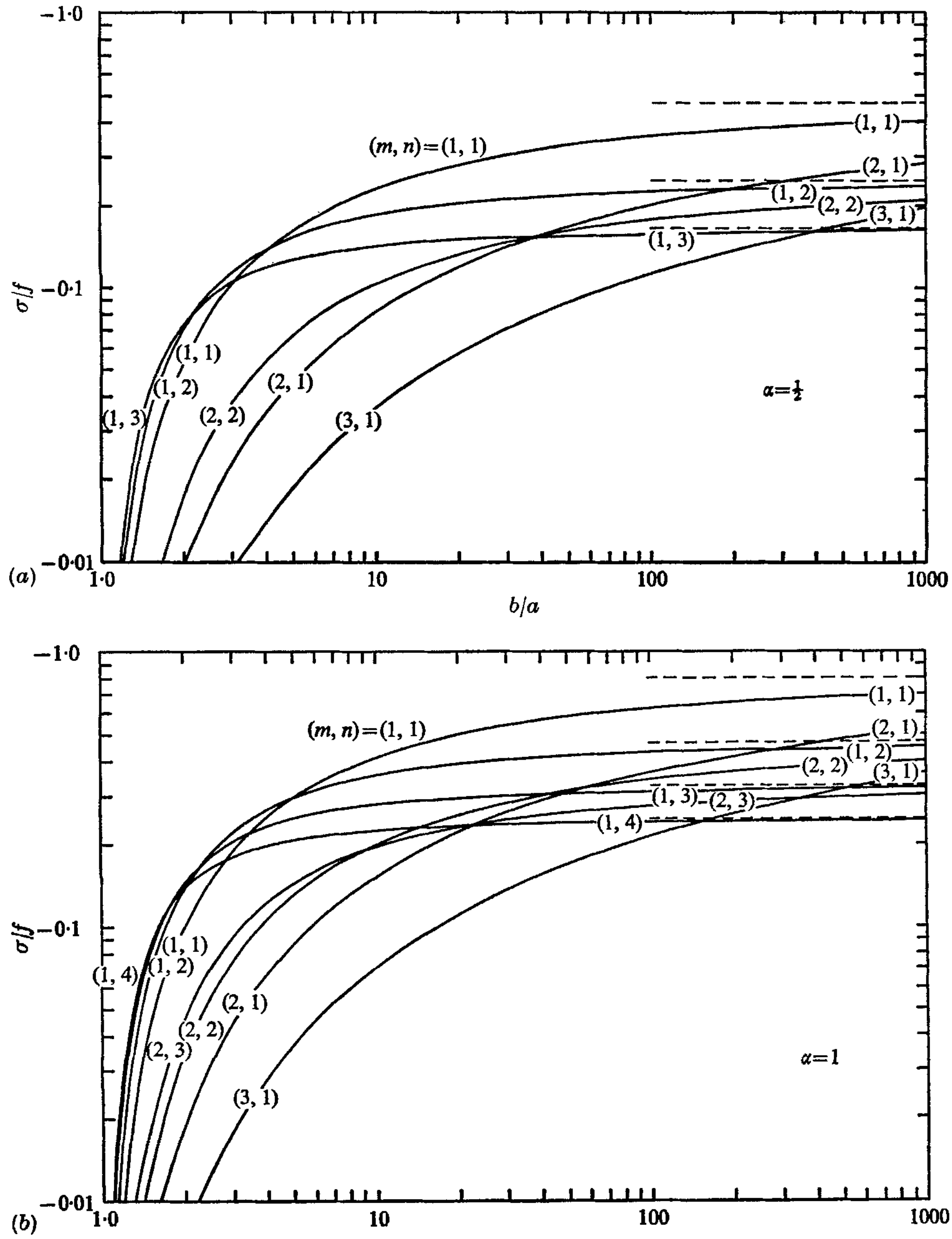

FIGURE 5 (*a*, *b*). For legend see p. 716.

In the important case $n = 1$, the total Stokes transport is given by the difference between $\rho\psi_s$ at infinity and ψ_s at $r = a$. Since ψ_s vanishes at $r = a$, and when $r > b$,

$$\hat{\psi}\hat{\psi}^* = CC^*r^2 + (CQ^* + C^*Q) + QQ^*r^{-2}, \tag{5.25}$$

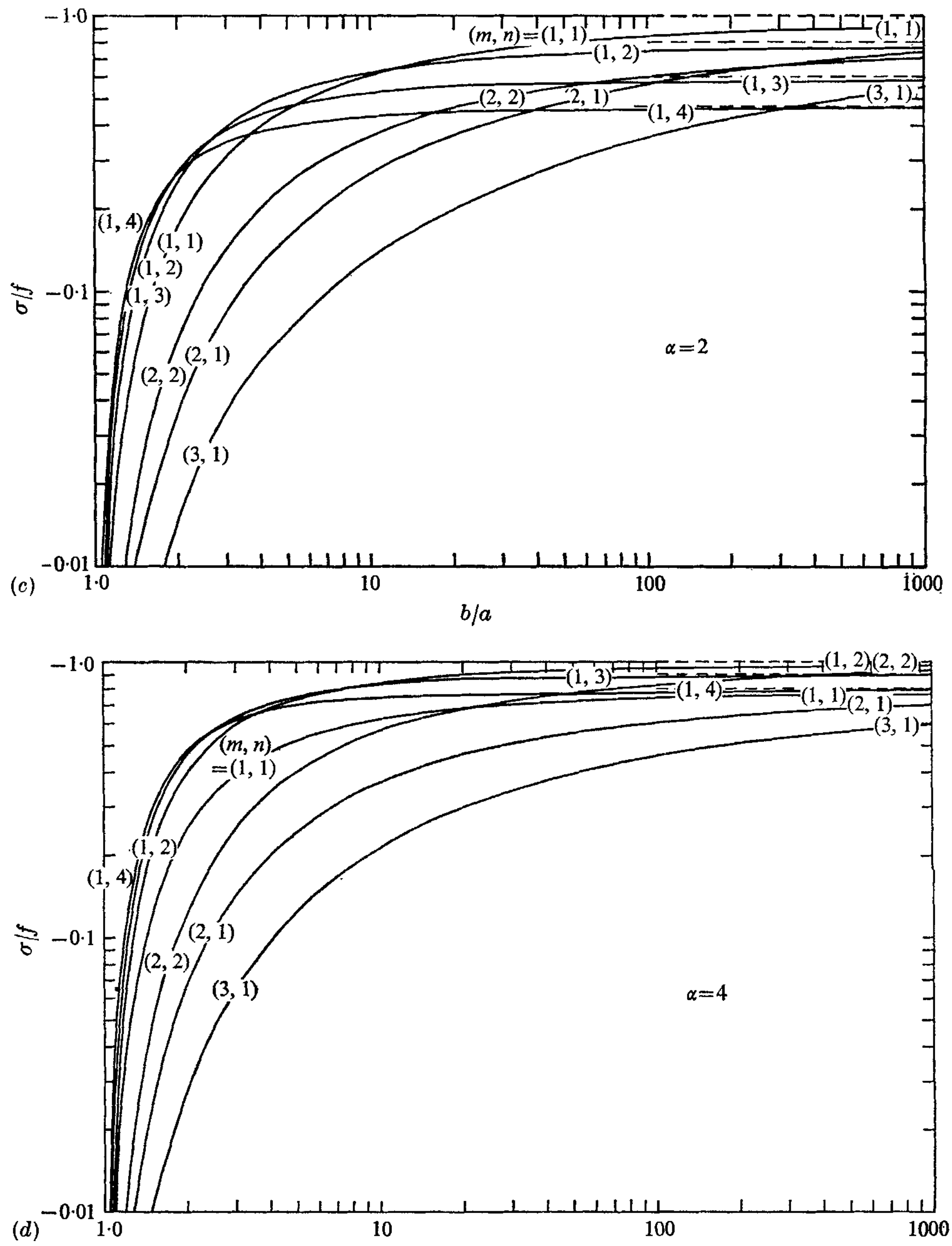

FIGURE 5. Graphs of the non-dimensional frequency (σ/f) for some of the lower modes, plotted as a function of b/a. The symbol (m, n) denotes the mth mode with azimuthal wave-number n. (a) $\alpha = \frac{1}{2}$, (b) $\alpha = 1$, (c) $\alpha = 2$, (d) $\alpha = 4$.

it follows from (5.22) that, as $r \to \infty$,

$$\psi_s \to CC^*/2\sigma. \tag{5.26}$$

This quantity has always the same sign as σ.

The *angular momentum* of the Stokes flow, on the other hand, is given by

$$\text{a.m.} = 2\pi\rho \int_a^\infty \frac{\partial \psi_s}{\partial r} r^2 \, dr. \tag{5.27}$$

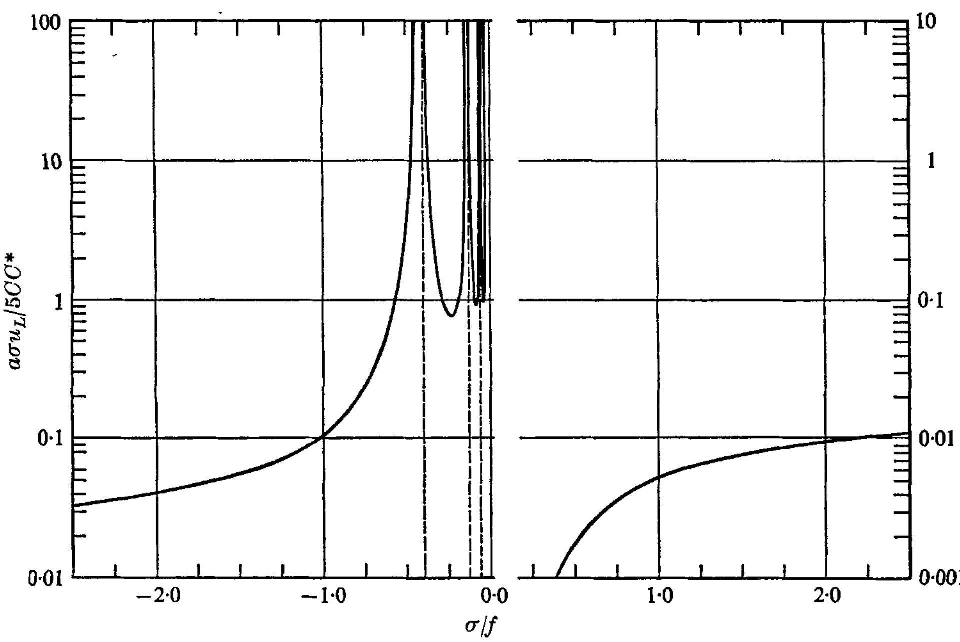

FIGURE 6. Relative magnitude of the mass-transport velocity near the perimeter of the island in a typical case: $\alpha = 2$, $b/a = 4$, showing the peaks at the resonant frequencies. The vertical scale on the right ($\sigma/f > 0$) is ten times that on the left.

In this expression ψ_s may be replaced by $(\psi_s - \psi_{s\infty})$, where $\psi_{s\infty}$ denotes the limit in (5.26). On integrating by parts, and using (5.22) and (5.24), we find

$$\text{a.m.} = -\frac{\pi\rho}{\sigma}(CQ^* + C^*Q). \tag{5.28}$$

This quantity may be either positive or negative, depending on the sign of Q. When the sides are vertical, we find from §3 that $Q = -Ca^2$ and so

$$\text{a.m.} = \frac{2\pi\rho a^2 CC^*}{\sigma}, \tag{5.29}$$

which has always the same sign as σ.

Consider now the Lagrangian mass-transport velocity u_L. This will depend on the viscous boundary conditions on the bottom and the vertical wall, and in general also on the time since the motion was started. But, on the vertical wall of the island, the velocity just outside the oscillatory boundary layer is given simply by (3.12), provided we now substitute $q_s = (-\partial\psi/\partial r)_{r=a}$. Since ψ is given by (5.4), we have then

$$u_L = \frac{5n}{4\sigma a}\left|\frac{d\hat{\psi}}{dr}\right|^2 = \frac{5n\beta^2P^2}{\sigma a^3}. \tag{5.30}$$

As before, the effect of viscosity is to multiply the mass-transport velocity near the perimeter of the island by the factor 5/2.

As an example let us take $\alpha = 2$, $b/a = 4$ and $n = 1$. Then figure 5(*c*) indicates that we must expect resonance when $\sigma/f = -0{\cdot}396$, $-0{\cdot}125$ and $-0{\cdot}056$, corresponding to the three lowest modes, $m = 1$, 2 and 3 respectively. In figure 6 we show a graph of the non-dimensional quantity,

$$\frac{n}{5}\frac{a\sigma u_L}{CC^*} = |\beta P|^2. \tag{5.31}$$

This represents the relative magnification of the mass-transport velocity near the perimeter $r = a$, compared to the mass transport in the absence of a surrounding 'skirt'. The amplification of the mass-transport near resonance can be clearly seen. By contrast, the response of the island to waves progressing in the clockwise direction ($\sigma/f > 0$) is remarkably small.

In calculating the amplitude of the forced oscillations, we have of course neglected the dissipation of energy by viscosity and also the detuning of the oscillations due to the slight dependence of the frequency on the amplitude. Both these effects will act to limit the amplitude near resonance.

6. Experimental verification (ii)

The following experiment provided a qualitative verification of the effects described in §5.

A circular tank of diameter 18 in. and depth about 6 in. was fitted with a false wax bottom in the form of a parabola, curved so as to be parallel to the free surface when rotating in equilibrium at a speed of 0·25 c/s. Projecting from the bottom of the tank were three islands. Two of these were circular cylinders with vertical sides and diameters 1 in. and $1\frac{1}{2}$ in. respectively (see figure 7). The third island was fitted with a 'skirt' corresponding to the parameters $\alpha = 2$, $b/a = 2$. The tank was placed on a rotating turntable and filled to a depth of about 3 in. (so as to cover the curved part of the skirted island). Aluminium powder was scattered on the surface to facilitate viewing the surface velocities.

The tank was then set in rotation at a speed of 0·25 c/s, and the relative motions were viewed through a rotascope.

On reaching the spin-up velocity one might have expected at first to see no relative motion between the fluid and the rotating tank. On the contrary, small oscillations, having the same period as the rotation, were observed in the main body of the fluid, due to the fact that the axis of rotation was not perfectly aligned with the vertical. The rotation of the tank being in the positive (eastwards) sense, the effect was to produce a component of gravity rotating in the negative (westwards) sense relative to the rotating tank. The frequency σ of the oscillation was thus given by

$$\sigma/f = -\tfrac{1}{2}, \tag{6.1}$$

since f is equal to twice the angular rate of rotation.

Close to the islands, however, the most obvious feature of the motion was not the oscillatory motion so much as an intense d.c. component of flow directed anticlockwise round each island. The motion was more intense round the smaller of the two cylindrical islands. This is to be expected from (3.14), in which the mass transport velocity is inversely proportional to the radius a of the island.

The most intense current, however, was observed near the third island, the one with the skirt. A computation of the relative velocity (5.31) in the case $\alpha = 2$, $b/a = 2$ and $\sigma/f = -\frac{1}{2}$ shows that, for these values, the relative magnification is given by

$$\frac{a\sigma u_L}{5CC^*} = 4{\cdot}432. \tag{6.2}$$

It was not possible to measure the drift currents accurately in this experiment but the observations appeared to be consistent with the above ratio. Further experiments are at present in progress.

7. Discussion

Equation (3.14) indicates that the order of magnitude of the streaming velocity is inversely proportional to the radius a of the island. It follows that the smaller the island, the greater the streaming velocity, within the present approximations. Hence a quite small island may be, as it were, a useful probe for detecting oscillatory motions in the surrounding ocean.

It does not seem altogether fanciful to suggest that the drift velocities observed in the neighbourhood of Bermuda by Stommel (1954) may be partly attributed to mass-transport streaming associated with oscillations nearby. From figure 1 of Stommel's paper it appears that the particle tracks nearly all circulate that island in the clockwise sense, and with times comparable to 15 days, or about 30 tidal cycles. If this is to be comparable with $a^2/5b^2$, where a is the mean radius of the island of Bermuda we must have $b/a \sim 0{\cdot}08$, or, since $a \sim 5\,\text{km}$, the half tidal displacement b must be of order $0{\cdot}4\,\text{km}$. This is not inconsistent with what is known of the tidal currents in that area.

This investigation was begun at Oregon State University, Corvallis, under NSF Grant No. GA-1452 and completed at the National Institute of Oceanography, England. The experiments described in §6 were carried out with the assistance of Steve Wilcox at Oregon State University, using a rotating turntable and rotascope constructed at N.I.O. The experiments in §4 were first performed at my home in Cambridge, and subsequently at Wormley. John Simpson held the stopwatch.

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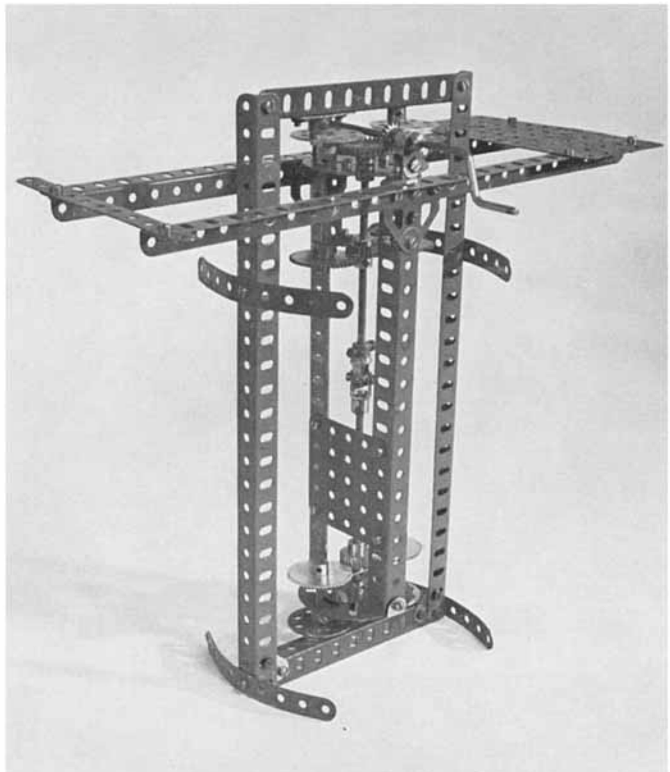
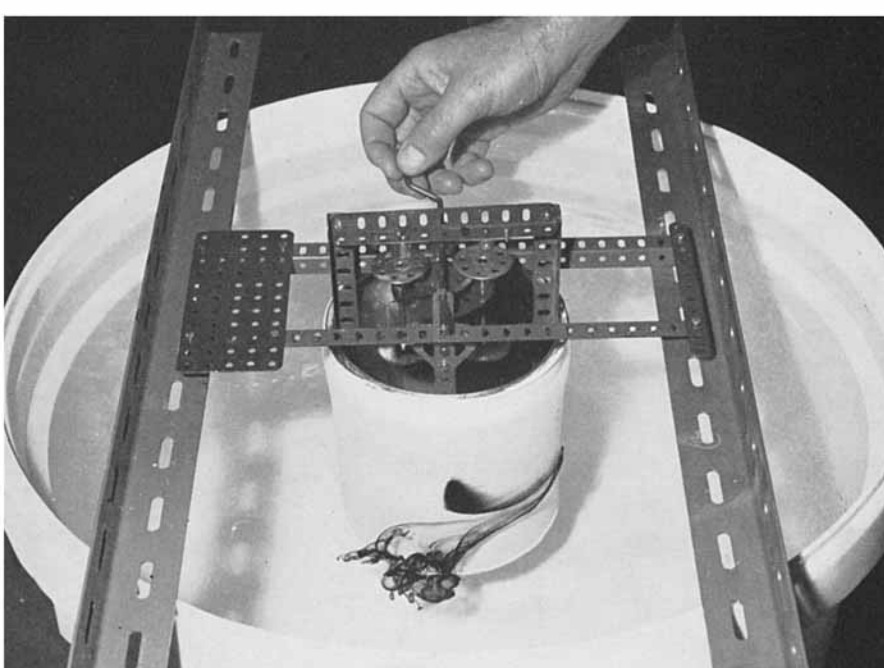


(*a*)



(*b*)

FIGURE 3. (*a*) A mechanism for oscillating a cylinder of radius 3 in. so that its axis describes a smaller cylinder of radius $\frac{1}{2}$ in. The orientation of the cylinder remains fixed.

(*b*) An experiment to measure the Lagrangian drift velocity near to the boundary of an oscillating cylinder, using the mechanism of figure 3(*a*).